

A splitting algorithm for stochastic partial differential equations driven by linear multiplicative noise

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Abstract

We study the convergence of a Douglas-Rachford type splitting algorithm for the infinite dimensional stochastic differential equation

$$dX + A(t)(X)dt = X dW \text{ in } (0, T); \quad X(0) = x,$$

where $A(t) : V \rightarrow V'$ is a nonlinear, monotone, coercive and demicontinuous operator with sublinear growth and V is a real Hilbert space with the dual V' . V is densely and continuously embedded in the Hilbert space H and W is an H -valued Wiener process. The general case of a maximal monotone operators $A(t) : H \rightarrow H$ is also investigated.

Keywords: Maximal monotone operator, stochastic process, parabolic stochastic equation.

Mathematics Subject Classification (2010): Primary 60H15; Secondary 47H05, 47J05.

1 Introduction

We consider here the stochastic differential equation

$$\begin{aligned} dX(t) + A(t)X(t)dt &= X(t)dW(t), \quad t \in (0, T), \\ X(0) &= x, \end{aligned} \tag{1.1}$$

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in a real separable Hilbert space H , whose elements are functions or distributions on a bounded and open set $\mathcal{O} \subset \mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$. In particular, H can be any of the spaces $L^2(\mathcal{O})$, $H_0^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$, $H^1(\mathcal{O})$, $k = 1, 2, \dots$, with the corresponding Hilbertian structure. Here $H_0^1(\mathcal{O})$, $H^k(\mathcal{O})$ are the standard L^2 -Sobolev spaces on \mathcal{O} , and W is a Wiener process of the form

$$W(t, \xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi) \beta_j(t), \quad \xi \in \mathcal{O}, \quad t \geq 0, \quad (1.2)$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of real-valued Brownian motions on a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ with natural filtration $(\mathcal{F}_t)_{t \geq 0}$. Here, $e_j \in C^2(\overline{\mathcal{O}}) \cap H$, $j \in \mathbb{N}$, is an orthonormal basis in H , and $\mu_j \in \mathbb{R}$, $j = 1, 2, \dots$.

The following hypotheses will be in effect throughout this work.

- (i) There is a Hilbert space V with dual V' such that $V \subset H$, continuously and densely. Hence $V \subset H (\equiv H') \subset V'$ continuously and densely.
- (ii) $A : [0, T] \times V \times \Omega \rightarrow V'$ is *progressively measurable*, i.e., for every $t \in [0, T]$, this operator restricted to $[0, t] \times V \times \Omega$ is $\mathcal{B}([0, t]) \otimes \mathcal{B}(V) \otimes \mathcal{F}_t$ -measurable.
- (iii) There is $\delta \geq 0$ such that, for each $t \in [0, T]$, $\omega \in \Omega$, the operator $u \mapsto \delta u + A(t, \omega)u$ is monotone and *demicontinuous* (that is, strongly-weakly continuous) from V to V' .

Moreover, there are $\alpha_i, \gamma_i \in \mathbb{R}$, $i = 1, 2, 3$, $\alpha_1 > 0$, such that, \mathbb{P} -a.s.,

$$\langle A(t, \omega)u, u \rangle \geq \alpha_1 |u|_V^2 + \alpha_2 |u|_H^2 + \alpha_3, \quad \forall u \in V, \quad t \in [0, T], \quad (1.3)$$

$$|A(t, \omega)u|_{V'} \leq \gamma_1 |u|_V + \gamma_2, \quad \forall u \in V, \quad t \in [0, T]. \quad (1.4)$$

- (iv) $e^{\pm W(t)}$ is, for each t , a multiplier in V and a multiplier in H such that there exists an (\mathcal{F}_t) -adapted, \mathbb{R}_+ -valued process $Z(t)$, $t \in [0, T]$, with

$$\mathbb{E} \left[\sup_{t \in [0, T]} |Z(t)| \right] < \infty \text{ for all } r \in [1, \infty) \text{ and such that, } \mathbb{P}\text{-a.s.,}$$

$$\begin{aligned} |e^{\pm W(t)}y|_V &\leq Z(t)|y|_V, \quad \forall t \in [0, T], \quad \forall y \in V, \\ |e^{\pm W(t)}y|_H &\leq Z(t)|y|_H, \quad \forall t \in [0, T], \quad \forall y \in H. \end{aligned} \quad (1.5)$$

One assumes also that, for each $\omega \in \Omega$, the function $t \rightarrow e^{\pm W(t)}$ is H -valued continuous on $[0, T]$.

Throughout in the following, $|\cdot|_V$ and $|\cdot|_{V'}$ denote the norms of V and V' , respectively, and by $\langle \cdot, \cdot \rangle$ we denote the duality pairing between V and V' with H as pivot space; on $H \times H$, $\langle \cdot, \cdot \rangle$ is just the scalar product of H . The norms of V and V' are denoted by $|\cdot|_H$ and $|\cdot|_V$, $|\cdot|_{V'}$, respectively, $\mathcal{B}(H)$, $\mathcal{B}(V)$ etc. are the classes of Borel sets in the corresponding spaces.

As regards the orthonormal basis $\{e_j\}_{j=1}^\infty$ in (1.2), we assume that there exist $\tilde{\gamma}_j \in [1, \infty)$ such that

$$|ye_j|_H \leq \tilde{\gamma}_j |y|_H, \quad \forall y \in H, \quad j = 1, 2, \dots, \quad \nu := \sum_{j=1}^\infty \mu_j^2 \tilde{\gamma}_j^2 |e_j|_\infty^2 < \infty. \quad (1.6)$$

and we assume also that

$$\mu := \frac{1}{2} \sum_{j=1}^\infty \mu_j^2 e_j^2 \quad (1.7)$$

is a multiplier in V , V' and H .

It should be noted that $X dW = \sigma(X) d\widetilde{W}$ where $\sigma : H \rightarrow L_2(H)$ (the space of Hilbert-Schmidt operators on H) is defined by

$$\sigma(u)v = \sum_{j=1}^\infty \mu_j u \langle v, e_j \rangle e_j, \quad \forall v \in H,$$

and so, $\widetilde{W} = \sum_{j=1}^\infty e_j \beta_j$ is a cylindrical Wiener process on H (see [5]).

Definition 1.1. By a *solution* to (1.1) for $x \in H$, we mean an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process $X : [0, T] \rightarrow H$ with continuous sample paths which satisfies

$$X(t) + \int_0^t A(s)X(s)ds = x + \int_0^t X(s)dW(s), \quad t \in [0, T], \quad (1.8)$$

$$X \in L^2((0, T) \times \Omega; V). \quad (1.9)$$

The stochastic integral arising in (1.8) is considered in Itô's sense.

In [3], the authors developed an operatorial approach to (1.1) under the more general hypotheses than (i)–(iv) above. As a special case (see Theorem 3.1 in [3]), we have

Theorem 1.2. *Under Hypotheses (i)–(iv), for each $x \in H$, equation (1.1) has a unique solution X (in the sense of Definition 1.1). Moreover, the function $t \mapsto e^{-W(t)}X(t)$ is V' -absolutely continuous on $[0, T]$ and*

$$\mathbb{E} \int_0^T \left| e^{W(t)} \frac{d}{dt} (e^{-W(t)}X(t)) \right|_{V'}^2 dt < \infty. \quad (1.10)$$

In a few words, the method developed in [3] is the following. By the transformation

$$X(t) = e^{W(t)}y(t), \quad t \geq 0, \quad (1.11)$$

one reduces equation (1.1) to the random differential equation

$$\begin{aligned} \frac{dy}{dt}(t) + e^{-W(t)}A(t)(e^{W(t)}y(t)) + \mu y(t) &= 0, \quad \text{a.e. } t \in (0, T), \\ y(0) &= x, \end{aligned} \quad (1.12)$$

and treat (1.12) as an operatorial equation of the form

$$\mathcal{B}y + \mathcal{A}y = 0 \quad (1.13)$$

in a suitable Hilbert space \mathcal{H} of stochastic processes on $[0, T]$. Here, \mathcal{A} and \mathcal{B} are maximal monotone operators suitable defined from \mathcal{V} to \mathcal{V}' , where $(\mathcal{V}, \mathcal{V}')$ is a dual pair of spaces such that $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ with dense and continuous embeddings.

The operatorial form (1.13) of equation (1.12) suggests to approximate the solution y by the Douglas–Rachford splitting algorithm ([6]–[8]).

The exact form and convergence of the corresponding splitting algorithm for equation (1.13) will be given below in Section 2. As seen later on in Theorem 2.1, it leads to a convergent splitting algorithm for the stochastic differential equation (1.1).

In this way, the operator theoretic approach to equation (1.1) written in the form (1.13) allows to design a convergent splitting scheme for equation (1.1) inspired by the Rockafellar [9] proximal point algorithm for nonlinear operatorial equations (on these lines see also [4]). By our knowledge, the splitting algorithm obtained here for the stochastic equation is new and might have implications in numerical approximation of stochastic PDEs.

Notations. If U is a Banach space, we denote by $L^p(0, T; U)$, $1 \leq p \leq \infty$, the space of all L^p -integrable U -valued functions on $(0, T)$. The space

$L^p((0, T) \times \Omega; U)$ is defined similarly. We refer to [2] for notation and standard results of the theory of maximal monotone operators in Banach spaces. If \mathcal{O} is an open domain of \mathbb{R}^d , we denote by $W^{1,p}(\mathcal{O})$, $1 \leq p \leq \infty$ and $H^1(\mathcal{O}), H^{-1}(\mathcal{O})$ the standard Sobolev spaces on \mathcal{O} .

2 Main results

Without loss of generality, we may assume that, besides assumptions (i)–(iii), $A(t)$ satisfies also the strong monotonicity condition

$$\langle A(t)u - A(t)v, u - v \rangle \geq \nu |u - v|_H^2, \quad \forall u, v \in V, \quad (2.1)$$

where $\nu > 0$ is given by (1.6). (In fact, as easily seen, by the substitution $X \rightarrow \exp(-(\nu + \delta)t)X$ with a suitable δ , equation (1.1) can be rewritten as

$$dX + \tilde{A}(t)X dt = X dW,$$

where the operator $X \rightarrow \tilde{A}(t)X = e^{-(\nu+\delta)t}A(t)(e^{(\nu+\delta)t}X) + (\nu + \delta)X$ satisfies conditions (i)–(iii) and (2.1).)

We associate with equation (1.1) the following splitting algorithm

$$\begin{aligned} \lambda dZ_{n+1} + J(Z_{n+1})dt + \lambda \nu Z_{n+1}dt &= \lambda Z_{n+1}dW - \lambda A(t)X_n dt \\ &+ \lambda \nu X_n dt + J(X_n)dt, \quad t \in (0, T), \end{aligned} \quad (2.2)$$

$$Z_{n+1}(0) = x, \quad n = 0, 1, \dots$$

$$\begin{aligned} \lambda A(t)X_{n+1}(t) + J(X_{n+1}(t)) - \lambda \nu X_{n+1}(t) \\ = J(Z_{n+1}(t)) + \lambda A(t)X_n(t) - \lambda \nu X_n(t), \end{aligned} \quad (2.3)$$

where $X_0 \in L^2((0, T) \times \Omega; V)$ is $(\mathcal{F}_t)_{t \geq 0}$ -adapted and arbitrary. Here, the parameter $\lambda > 0$ is arbitrary but fixed and $J : V \rightarrow V'$ is the canonical isomorphism of the space V onto its dual V' .

Taking into account assumptions (i)–(iii) and (2.1), which, in particular, implies that the operator $\Gamma_0 : L^2(0, T; V) \rightarrow L^2(0, T; V')$, $\Gamma_0 u = \lambda A(t)u + J(u) - \lambda \nu u$, $u \in L^2(0, T; V)$, is demicontinuous, locally bounded, and with inverse continuous, we see that the sequence (Z_n, X_n) is well defined by (2.2), (2.3) and we have also

$$X_n, Z_n \in L^2((0, T) \times \Omega; V) \text{ and } Z_n \in L^2(\Omega; C([0, T]; H)), \quad n = 1, 2, \dots \quad (2.4)$$

Moreover, the processes X_n, Z_n are $(\mathcal{F}_t)_{t \geq 0}$ -adapted on $[0, T]$.

Theorem 2.1 is the main result.

Theorem 2.1. *Under Hypotheses (i)–(iv) and (2.1), assume that $x \in V$ and $\lambda > 0$. If (X_n, Z_n) is the sequence defined by (2.2), (2.3), we have for $n \rightarrow \infty$*

$$X_n \rightarrow X \text{ weakly in } L^2((0, T) \times \Omega; V), \quad (2.5)$$

where X is the solution to equation (1.1) given by Theorem 1.2. Assume further that the operator $u \rightarrow A(t)u$ is odd, that is, $A(t)(-u) = -A(t)u$, $\forall u \in V$. Then, for $n \rightarrow \infty$,

$$X_n \rightarrow X \text{ strongly in } L^2((0, T) \times \Omega; V). \quad (2.6)$$

The splitting scheme (2.2)–(2.3) reduces the approximation of problem (1.1) to a sequence of simpler linear equations. In fact, at each step n , one should solve a linear stochastic differential equation of the form

$$\begin{aligned} dZ_{n+1} + \frac{1}{\lambda} J(Z_{n+1})dt + \nu Z_{n+1}dt &= Z_{n+1}dW + F_n dt, \quad t \in (0, T), \\ Z_{n+1}(0) &= x, \end{aligned} \quad (2.7)$$

and the stationary random equation (2.3), where

$$F_n = -\lambda A(t)X_n + \lambda \nu X_n + J(X_n).$$

By Itô's formula (see, e.g., [3]), equation (2.7) has, for each n , the solution

$$Z_{n+1} = e^W z_{n+1},$$

where z_{n+1} is the solution to the random differential equation

$$\begin{aligned} \frac{d}{dt} z_{n+1} + \frac{1}{\lambda} e^{-W} J(e^W z_{n+1}) + (\mu + \nu)z_{n+1} &= e^{-W} F_n, \\ z_{n+1}(0) &= x. \end{aligned} \quad (2.8)$$

If $F : L^2((0, T) \times \Omega; V') \rightarrow L^2((0, T) \times \Omega; V)$ is the linear continuous operator defined by

$$F(f) = Y,$$

where Y is the solution to the stochastic equation

$$dY + \frac{1}{\lambda} J(Y)dt + \nu Y dt = Y dW + f dt; \quad Y(0) = x,$$

then we may rewrite (2.2)–(2.3) as

$$X_{n+1} = (\lambda(A - \nu I) + J)^{-1} [JF((\lambda(\nu I - A) + J)(X_n)) + \lambda(A - \nu I)X_n], \quad n = 0, 1, \dots$$

Equivalently,

$$X_{n+1} = \Gamma^n X_0, \quad \forall n \in \mathbb{N}, \quad (2.9)$$

where $\Gamma : L^2((0, T) \times \Omega; V) \rightarrow L^2((0, T) \times \Omega; V)$ is the Lipschitzian and given by

$$\Gamma = (\lambda(A - \nu I) + J)^{-1} [JF(\lambda(\nu I - A) + J) + \lambda(A - \nu I)]. \quad (2.10)$$

Then, by Theorem 2.1, we get

Corollary 2.2. *Under assumptions (i)–(iv), (2.1), for each $\lambda > 0$ the solution X to (1.1) is expressed as*

$$X = w - \lim_{n \rightarrow \infty} \Gamma^n X_0 \quad \text{in } L^2((0, T) \times \Omega; V), \quad (2.11)$$

where $X_0 \in L^2((0, T) \times \Omega; V)$ is an arbitrary $(\mathcal{F}_t)_{t \geq 0}$ -adapted process.

Here $w - \lim$ indicates the weak limit.

3 Proof of Theorem 2.1

Proceeding as in [3], we consider the spaces \mathcal{H} , \mathcal{V} and \mathcal{V}' , defined as follows. \mathcal{H} is the Hilbert space of all $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes $y : [0, T] \rightarrow H$ such that

$$|y|_{\mathcal{H}} = \left(\mathbb{E} \int_0^T |e^{W(t)} y(t)|_H^2 dt \right)^{\frac{1}{2}} < \infty,$$

where \mathbb{E} denotes the expectation in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The space \mathcal{H} is endowed with the norm $|\cdot|_{\mathcal{H}}$ generated by the scalar product

$$\langle y, z \rangle_H = \mathbb{E} \int_0^T \langle e^{W(t)} y(t), e^{E(t)} y(t) \rangle dt.$$

\mathcal{V} is the space of all $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes $y : [0, T] \rightarrow V$ such that

$$|y|_{\mathcal{V}} = \left(\mathbb{E} \int_0^T |e^{W(t)} y(t)|_V^2 dt \right)^{\frac{1}{2}} < \infty.$$

\mathcal{V}' (the dual of \mathcal{V}) is the space of all $(\mathcal{F}_t)_{t \geq 0}$ -adapted processes $y : [0, T] \rightarrow V'$ such that

$$|y|_{\mathcal{V}'} = \left(\mathbb{E} \int_0^T |e^{W(t)} y(t)|_{V'}^2 dt \right)^{\frac{1}{2}} < \infty.$$

We have $\mathcal{V} \subset \mathcal{H} \subset \mathcal{V}'$ with continuous and dense embeddings. Moreover,

$${}_{\mathcal{V}'} \langle u, v \rangle_{\mathcal{V}} = \mathbb{E} \int_0^T \langle e^{W(t)} u(t), e^{W(t)} v(t) \rangle dt, \quad v \in \mathcal{V}, \quad u \in \mathcal{V}',$$

is the duality pairing between \mathcal{V} and \mathcal{V}' , with the pivot space \mathcal{H} , that is,

$${}_{\mathcal{V}'} \langle u, v \rangle_{\mathcal{V}} = \langle u, v \rangle_{\mathcal{H}}, \quad \forall u \in \mathcal{H}, \quad v \in \mathcal{V}.$$

Now, for $x \in H$, define the operators $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}'$ and $\mathcal{B} : D(\mathcal{B}) \subset \mathcal{V} \rightarrow \mathcal{V}'$ as follows:

$$\begin{aligned} (\mathcal{A}y)(t) &= e^{-W(t)} A(t)(e^{W(t)} y(t)) - \nu y(t), \quad \text{a.e. } t \in (0, T), \quad y \in \mathcal{V}, \\ (\mathcal{B}y)(t) &= \frac{dy}{dt}(t) + (\mu + \nu)y(t), \quad \text{a.e. } t \in (0, T), \quad y \in D(\mathcal{B}), \end{aligned} \quad (3.1)$$

$$D(\mathcal{B}) = \left\{ y \in \mathcal{V} : y \in AC([0, T]; V') \cap C([0, T]; H), \quad \mathbb{P}\text{-a.s.}, \quad \frac{dy}{dt} \in \mathcal{V}', \quad y(0) = x \right\}. \quad (3.2)$$

Here, $AC([0, T]; V')$ is the space of all absolutely continuous V' -valued functions on $[0, T]$. If $y \in D(\mathcal{B})$, then $y \in C([0, T]; H)$ and $\frac{dy}{dt}$ is the derivative of y in the sense of V' -valued distributions on $(0, T)$. Then, equation (1.12) can be expressed as

$$\mathcal{B}y + \mathcal{A}y = 0. \quad (3.3)$$

Then, the map $\Lambda : \mathcal{V} \rightarrow \mathcal{V}'$ defined by

$$\Lambda v = e^{-W} J(e^W v), \quad v \in V, \quad (3.4)$$

is the canonical isomorphism of \mathcal{V} onto \mathcal{V}' and the scalar product ${}_{\mathcal{V}} \langle \cdot, \cdot \rangle_{\mathcal{V}}$ of the space \mathcal{V} can be expressed as

$${}_{\mathcal{V}} \langle v, \bar{v} \rangle_{\mathcal{V}} = {}_{\mathcal{V}} \langle v, \Lambda \bar{v} \rangle_{\mathcal{V}'}, \quad \forall v, \bar{v} \in \mathcal{V}. \quad (3.5)$$

We set

$$(\mathcal{A}^*u)(t) = \Lambda^{-1}\mathcal{A}u(t) = e^{-W}J^{-1}(A(t)(e^W u) - \nu e^W u), \forall u \in \mathcal{V}, \quad (3.6)$$

$$(\mathcal{B}^*u)(t) = \Lambda^{-1}\mathcal{B}u(t) = e^{-W}J^{-1}\left(e^W\left(\frac{du}{dt} + (\mu + \nu)u\right)\right), \quad (3.7)$$

$$\forall u \in D(\mathcal{B}^*) = D(\mathcal{B}).$$

Since the operators \mathcal{A} , \mathcal{B} and $\mathcal{A} + \mathcal{B}$ are maximal monotone in $\mathcal{V} \times \mathcal{V}'$ ([3], Lemma 4.1, Lemma 4.2), it is easily seen by (3.6)-(3.7) that \mathcal{A}^* , \mathcal{B}^* and $\mathcal{A}^* + \mathcal{B}^*$ are maximal monotone in $\mathcal{V} \times \mathcal{V}$.

On the other hand, by (3.3) we can rewrite equation (3.3) as

$$\mathcal{B}^*y + \mathcal{A}^*y = 0. \quad (3.8)$$

Let $y \in D(\mathcal{B})$ be the unique solution to equation (3.3) (see [3], Proposition 3.3). Then, y is also the solution to (3.8) and so, by Theorem 1 in [8] (see, also, Corollary 6.1 in [7]), we have that

$$y = \lim_{n \rightarrow \infty} (I + \lambda \mathcal{A}^*)^{-1}v_n \text{ weakly in } \mathcal{V} \text{ as } n \rightarrow \infty, \quad (3.9)$$

where $\{v_n\} \subset \mathcal{V}$ is, for $n \geq 0$, defined by

$$v_{n+1} = (I + \lambda \mathcal{B}^*)^{-1}(2(I + \lambda \mathcal{A}^*)^{-1}v_n - v_n) + (I - (I + \lambda \mathcal{A}^*)^{-1})v_n, \quad (3.10)$$

and v_0 is arbitrary in \mathcal{V} . Here, I is the identity operator in \mathcal{V} .

The splitting algorithm (3.9)–(3.10) is just the Douglas–Rachford algorithm ([6]) for equation (3.8) and it can be equivalently expressed as

$$y = \lim_{n \rightarrow \infty} y_n \text{ weakly in } \mathcal{V}, \quad (3.11)$$

$$y_n = (I + \lambda \mathcal{A}^*)^{-1}v_n, \quad n = 0, 1, \dots, \quad (3.12)$$

$$y_{n+1} + \lambda \mathcal{A}^*y_{n+1} = z_{n+1} + v_n - y_n, \quad (3.13)$$

$$z_{n+1} + \lambda \mathcal{B}^*z_{n+1} = 2y_n - v_n, \quad (3.14)$$

where $v_0 \in \mathcal{V}$. (To get (3.12)–(3.14) from (3.10), we have used the identity $(I + \lambda \mathcal{B}^*)^{-1}(v + \lambda \mathcal{B}^*v) = v$, $\forall v \in D(\mathcal{B}^*)$ and the linearity of \mathcal{B}^* .)

In fact, the weak convergence of $\{v_n\}$ in the space \mathcal{V} is also a consequence of the convergence of the Rockafellar proximal point algorithm [9] for the maximal monotone operator $v \rightarrow G^{-1}(v) - v$, where

$$G(z) = (I + \lambda \mathcal{B}^*)^{-1}(2(I + \lambda \mathcal{A}^*)^{-1}z - z) + z - (I + \lambda \mathcal{A}^*)^{-1}z, \quad \forall z \in \mathcal{V}. \quad (3.15)$$

(See [7], Theorem 4.) Taking into account (3.6), (3.7), (3.12) we rewrite (3.14) as

$$\begin{aligned}
& e^{-W} J(e^W z_{n+1}) + \lambda \left(\frac{dz_{n+1}}{dt} + (\mu + \nu) z_{n+1} \right) \\
&= e^{-W} J(e^W (2y_n - v_n)) = e^{-W} J(e^W (-\lambda \mathcal{A}^* y_n + y_n)) \\
&= -\lambda e^{-W} A(t)(e^W y_n) + \lambda \nu y_n + e^{-W} J(e^W y_n)
\end{aligned} \tag{3.16}$$

and (3.13) as

$$\begin{aligned}
& J(e^W y_{n+1}) + \lambda A(t)(e^W y_{n+1}) - \lambda \nu e^W y_{n+1} \\
&= J(e^W (z_{n+1} + v_n - y_n)).
\end{aligned} \tag{3.17}$$

We set

$$X_n = e^W y_n, \quad Z_n = e^W z_n.$$

Then, by (3.16), we get via Itô's formula (see [3] and (2.8), (2.7))

$$\begin{aligned}
& \lambda dZ_{n+1} + J(Z_{n+1})dt + \lambda \nu Z_{n+1}dt = \lambda Z_{n+1}dW - \lambda A(t)X_n dt \\
& \quad + \lambda \nu X_n dt + J(X_n)dt, \\
& Z_{n+1}(0) = x.
\end{aligned}$$

By (3.17) and (3.12), we also get that

$$\begin{aligned}
& \lambda A(t)X_{n+1}(t) + J(X_{n+1}(t)) - \lambda \nu X_{n+1}(t) \\
&= J(Z_{n+1}(t)) + \lambda A(t)X_n(t) - \lambda \nu X_n(t), \quad t \in (0, T),
\end{aligned}$$

which are just equations (2.2), (2.3). Moreover, by (3.11), we see that (2.5) holds.

Assume now that $A(t) : V \rightarrow V'$ is odd. Then so is $\mathcal{A}^* : \mathcal{V} \rightarrow \mathcal{V}$ and also the operator G defined by (3.15). Then, according to a result of J. Baillon [1], the sequence $\{v_n\}$ defined by (3.10), that is $v_{n+1} = G(v_n)$, is strongly convergent in \mathcal{V} . Recalling (3.9), we infer that so is the sequence $\{y_n\}$ and, consequently, (2.6) holds. This completes the proof of Theorem 2.1.

Remark 3.1. One might expect that a similar splitting scheme can be constructed for nonlinear monotone operators $A(t) : V \rightarrow V'$, where V is a

reflexive Banach space and $A(t)$ are demicontinuous coercive and with polynomial growth as in [3]. In fact, in this case, one might replace (2.2) by

$$\begin{aligned}\lambda dZ_{n+1} + Z_{n+1}dt + \lambda\nu Z_{n+1}dt &= Z_{n+1}dW - \lambda A_H(t)X_n dt + \lambda\nu X_n dt + X_n dt, \\ t &\in (0, T), \\ \lambda A_H(t)X_{n+1} + X_{n+1} - \lambda\nu X_{n+1} &= Z_{n+1} + \lambda A_H(t)X_n - \lambda\nu X_n,\end{aligned}$$

where $A_H(t)u = A(t)u \cap H$. This question will be addressed in Section 5 below (see Remark 5.2).

4 Examples

We shall illustrate here the splitting algorithm (2.2)–(2.3) for a few parabolic stochastic differential equations.

Example 4.1. *Nonlinear stochastic parabolic equations.*

Consider the reaction-diffusion stochastic equation in $\mathcal{O} \subset \mathbb{R}^d$,

$$\begin{aligned}dX - \operatorname{div}(a(t, \xi, \nabla X))dt + \nu X dt + \psi(X)dt &= X dW \text{ in } (0, T) \times \mathcal{O}, \\ X &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \quad X(0) = x \text{ in } \mathcal{O}.\end{aligned}\tag{4.1}$$

Here, $a : (0, T) \times \mathcal{O} \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable in (t, ξ, r) continuous in r on \mathbb{R}^d , $a(t, \xi, 0) = 0$. (The more general case, when $a : (0, T) \times \mathcal{O} \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is progressively measurable, could also be considered.) We assume also that

$$\begin{aligned}(a(t, \xi, r_1) - a(t, \xi, r_2)) \cdot (r_1 - r_2) &\geq 0, \quad \forall r_1, r_2 \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O}, \\ a(t, \xi, r) \cdot r &\geq a_1 |r|_d^2 + a_2, \quad \forall r \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O}, \\ |a(t, \xi, r)|_d &\leq c_1 |r|_d + c_2, \quad \forall r \in \mathbb{R}^d, \quad (t, \xi) \in (0, T) \times \mathcal{O},\end{aligned}$$

where $a_1, c_1, \nu > 0$, $a_2, c_2 \in \mathbb{R}$, are independent of (t, ξ) , and $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous and monotonically nondecreasing function such that $\psi(0) = 0$ and $|\psi(r)| \leq C(|r|^{\frac{2d}{d+2}} + 1)$, $\forall r \in \mathbb{R}$. Here $\mathcal{O} \subset \mathbb{R}^d$ is a bounded open subset with smooth boundary $\partial\mathcal{O}$, and $|\cdot|_d$ is the Euclidean norm of \mathbb{R}^d .

If $H = L^2(\mathcal{O})$, $V = H_0^1(\mathcal{O})$, $V' = H^{-1}(\mathcal{O})$ and , for $t \in (0, T)$, the operator $A(t) : V \rightarrow V'$ is defined by

$${}_{V'} \langle A(t)y, \varphi \rangle_V = \int_{\mathcal{O}} (a(t, \xi, \nabla y) \cdot \nabla \varphi + \psi(y)\varphi) d\xi, \quad \forall \varphi \in H_0^1(\mathcal{O}), \quad y \in H_0^1(\mathcal{O}),$$

then Hypotheses (i)–(iii) are satisfied. As regards the Wiener process W , we assume here that, besides (1.6), the following condition holds:

$$\sum_{j=1}^{\infty} \mu_j^2 |\nabla e_j|_{\infty}^2 < \infty.$$

Then, by Theorem 2.1, where H , V and $A(t)$ are defined above and $J = -\Delta$ with Dirichlet homogeneous boundary conditions, if $x \in H_0^1(\mathcal{O})$, the solution

$$X \in L^2(\Omega; C([0, T]; L^2(\mathcal{O})) \cap L^2((0, T) \times \Omega; H_0^1(\mathcal{O})))$$

to (4.1) can be obtained as

$$X = w - \lim_{n \rightarrow \infty} X_n \text{ in } L^2((0, T) \times \Omega; H_0^1(\mathcal{O})), \quad (4.2)$$

where $(X_n, Z_n) \in L^2((0, T) \times \Omega; H_0^1(\mathcal{O}))$ is the solution to the system (we take $\lambda = 1$)

$$\begin{aligned} dZ_{n+1} - \Delta Z_{n+1} dt + \nu Z_{n+1} dt &= Z_{n+1} dW + \operatorname{div}(a(t, \xi, \nabla X_n)) dt - \Delta X_n dt \\ &\text{in } (0, T) \times \mathcal{O}, \\ Z_{n+1}(0) &= x \text{ in } \mathcal{O}, \\ Z_{n+1} &= 0 \text{ in } (0, T) \times \partial\mathcal{O}, \\ \operatorname{div} a(\nabla X_{n+1}) + \Delta X_{n+1} &= \Delta Z_{n+1} + \operatorname{div}(a(t, \xi, \nabla X_n)) \text{ in } (0, T) \times \mathcal{O}, \end{aligned} \quad (4.3)$$

where $X_0 \in L^2((0, T) \times \Omega; H_0^1(\mathcal{O}))$ is arbitrary but \mathcal{F}_t -adapted. Moreover, if $a(t, \xi, -r) \equiv -a(t, \xi, r)$, $\forall r \in \mathbb{R}^d$, then the convergence (4.2) is strong in $L^2((0, T) \times \Omega; H_0^1(\mathcal{O}))$.

Example 4.2. *Stochastic porous media equations.*

Consider the stochastic equation

$$\begin{aligned} dX - \Delta \psi(t, \xi, X) dt - \nu \Delta X dt &= X dW \text{ in } (0, T) \times \mathcal{O}, \\ X(0, \xi) &= x(\xi) \text{ in } \mathcal{O}, \\ \psi(t, \xi, X(t, \xi)) &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \end{aligned} \quad (4.4)$$

where \mathcal{O} is a bounded domain in \mathbb{R}^d , $\nu > 0$, the function $\psi : [0, T] \times \overline{\mathcal{O}} \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous, $r \rightarrow \psi(t, \xi, r)$ is monotonically increasing in r , and there exist

$a \in (0, \infty)$ and $c \in [0, \infty)$ such that

$$\begin{aligned} r\psi(t, \xi, r) &\geq a|r|^2 - c, \quad \forall r \in \mathbb{R}, (t, \xi, r) \in [0, T] \times \overline{\mathcal{O}}, \\ |\psi(t, \xi, r)| &\leq c(1 + |r|), \quad \forall r \in \mathbb{R}, (t, \xi, r) \in [0, T] \times \overline{\mathcal{O}}. \end{aligned} \quad (4.5)$$

We shall write equation (4.4) under the form (1.1) with $H = H^{-1}(\mathcal{O})$. Namely, we take $V = L^2(\mathcal{O})$, $H = H^{-1}(\mathcal{O})$, and V' is the dual of V with the pivot space $H^{-1}(\mathcal{O})$. Then, $V \subset H \subset V'$ and

$$V' = \{\theta \in \mathcal{D}'(\mathcal{O}) : \theta = -\Delta v, v \in L^2(\mathcal{O})\},$$

where Δ is taken in the sense of distributions on \mathcal{O} . (Here $\mathcal{D}'(\mathcal{O})$ is the space of Schwartz distributions on \mathcal{O} .) The duality ${}_{V'} \langle \cdot, \cdot \rangle_V$ is defined as

$${}_{V'} \langle \theta, u \rangle_V = \int_{\mathcal{O}} \tilde{\theta} u d\xi, \quad \tilde{\theta} = (-\Delta)^{-1} \theta,$$

where Δ is the Laplace operator with homogeneous Dirichlet boundary conditions on $\partial\mathcal{O}$. The duality mapping $J : V \rightarrow V'$ is just the operator $-\Delta$ defined from $L^2(\mathcal{O})$ to $V' \subset \mathcal{D}'(\mathcal{O})$ by

$$\Delta u(\varphi) = \int_{\mathcal{O}} u \Delta \varphi d\xi, \quad \forall \varphi \in H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}).$$

The operator $A(t) : V \rightarrow V'$ is defined by

$${}_{V'} \langle A(t)y, v \rangle_V = \int_{\mathcal{O}} \psi(t, \xi, y)v d\xi, \quad \forall y, v \in V = L^2(\mathcal{O}), t \in [0, T].$$

Then, Hypotheses (i)–(iv) hold and so, if $x \in L^2(\mathcal{O})$, by Theorem 2.1, the solution $X \in L^2(\Omega; C([0, T]; H^{-1}(\mathcal{O})) \cap L^2((0, T) \times \Omega; L^2(\mathcal{O})))$ to (4.4) is given by

$$X = w - \lim_{n \rightarrow \infty} X_n \text{ in } L^2((0, T) \times \Omega; L^2(\mathcal{O})),$$

where

$$\begin{aligned} dZ_{n+1} - \Delta Z_{n+1} dt + \nu Z_{n+1} dt &= Z_{n+1} dW - \Delta \psi(t, \cdot, X_n) dt - \Delta Z_n dt \\ &\text{in } (0, T) \times \mathcal{O}, \\ Z_{n+1}(0) &= x \in L^2(\mathcal{O}), \quad n = 0, 1, \dots, \\ \Delta \psi(t, \cdot, X_{n+1}) + X_{n+1} &= Z_{n+1} + \Delta \psi(t, \cdot, X_n), \quad \text{in } \mathcal{O}, \\ \psi(t, \cdot, X_{n+1}(t, \cdot)) &= 0 \quad \text{on } \partial\mathcal{O}, \\ n = 0, 1, \dots, X_0 &\in L^2(0, T; L^2(\Omega; L^2(\mathcal{O}))). \end{aligned} \quad (4.6)$$

If $\psi(t, \xi, r) = -\psi(t, \xi, -r)$, $\forall r \in \mathbb{R}$, then the convergence of the sequence $\{X_n\}$ is strong in $L^2((0, T) \times \Omega; L^2(\mathcal{O}))$.

5 The case where $A(t)$ is maximal monotone in $H \times H$

Consider now equation (1.1) under the following assumptions on A :

- (j) $A : [0, T] \times H \times \Omega \rightarrow H$ is progressively measurable and, for each $(t, \omega) \times [0, T] \times \Omega$ the operator $u \rightarrow A(t, \omega, u)$ is maximal monotone in $H \times H$. Moreover, there is $f \in L^2((0, T) \times \Omega; H)$ such that

$$(I + A(t))^{-1}f(t) \in L^2((0, T) \times \Omega; H). \quad (5.1)$$

We assume also that condition (2.1) holds.

It should be noted that, if $A(t) : V \rightarrow V'$ satisfies assumptions (i)-(ii), where V is a reflexive Banach space, then the operator $A(t) : H \rightarrow H$, defined by

$$A(t)_H u = A(t)u \cap V,$$

satisfies assumption (j). However, the class of the operators A satisfying (j) is considerably larger.

We consider the splitting scheme (which is well defined by strong monotonicity of $\mathcal{A}_1^* + \mathcal{B}_1^*$)

$$\begin{aligned} \lambda dY_{n+1} + (1 + \lambda\nu)Y_{n+1}dt &= \lambda Y_{n+1}dW \\ &+ (V_n - ((1 - \lambda\nu)I - \lambda A(t))^{-1}V_n)dt, \\ Y_{n+1}(0) &= x \text{ in } (0, T), \\ V_{n+1} &= Y_{n+1} + V_n - ((1 - \lambda\nu)I - \lambda A(t))^{-1}V_n, \end{aligned} \quad (5.2)$$

where $V_0 \in L^2((0, T) \times \Omega; H)$ is an $(\mathcal{F}_t)_{t \geq 0}$ -adapted process such that $A(t)V_0 \in L^2((0, T) \times \Omega; H)$. We have

Theorem 5.1. *Assume that $x \in H$ and that equation (1.1) has a solution $X \in L^2(\Omega; C([0, T]; H))$ such that $A(t)X \in L^2((0, T) \times \Omega; H)$.*

Then, for $n \rightarrow \infty$,

$$V_n \rightarrow V \text{ weakly in } L^2((0, T) \times \Omega; H), \quad (5.3)$$

where $X = ((1 - \lambda\nu)I + A(t))^{-1}V$ is the solution to (1.1).

If $A(t)$ is odd, then the convergence (5.3) is strong.

Proof. The operators \mathcal{A}_1^* and \mathcal{B}_1^* defined by

$$\begin{aligned} (\mathcal{A}_1^*u)(t) &= e^{-W}A(t)(e^Wu) - \nu u, \quad \forall u \in D(\mathcal{A}_1^*), \\ (\mathcal{B}_1^*u)(t) &= \frac{du}{dt} + (\mu + \nu)u, \quad \forall u \in D(\mathcal{B}_1^*), \end{aligned}$$

with the domains

$$\begin{aligned} D(\mathcal{A}_1^*) &= \{u \in \mathcal{H}; e^{-W}A(t)(e^Wu) - \nu u \in \mathcal{H}\}, \\ D(\mathcal{B}_1^*) &= \{u \in \mathcal{H}; u \in W^{1,2}([0, T]; H) \cdot \mathbb{P}\text{-a.s.}, u(0) = x\} \end{aligned}$$

are, by the above hypotheses, maximal monotone in $\mathcal{H} \times \mathcal{H}$ (see also [4]). Moreover, there is at least one solution y^* to the equation

$$\mathcal{A}_1^*y^* + \mathcal{B}_1^*y^* = 0. \quad (5.4)$$

Then, again by [8], it follows that the sequence $\{v_n\} \subset \mathcal{H}$ defined by

$$v_{n+1} = (I + \lambda\mathcal{B}_1^*)^{-1}(2(I + \lambda\mathcal{A}_1^*)^{-1}v_n - v_n) + v_n - (I + \lambda\mathcal{A}_1^*)^{-1}v_n, \quad n = 0, 1, \dots \quad (5.5)$$

is weakly convergent in \mathcal{H} to v^* , where $(1 + \lambda\mathcal{A}_1^*)^{-1}v^* = y^*$ is the solution to equation (5.4).

We set

$$\tilde{z}_{n+1} = v_{n+1} - v_n + (I + \lambda\mathcal{A}_1^*)^{-1}v_n \quad (5.6)$$

and, by (5.5), we have

$$\tilde{z}_{n+1} + \lambda\mathcal{B}_1^*\tilde{z}_{n+1} = v_n - (I + \lambda\mathcal{A}_1^*)^{-1}v_n. \quad (5.7)$$

Then, if $Y_n = e^W\tilde{z}_n$ and $V_n = e^Wv_n$, we can rewrite (5.6)-(5.7) as (5.2) and get (5.3), as claimed.

Remark 5.2. The convergence of the splitting algorithm (5.1)-(5.2) does not require conditions of the form (ii)-(iii) for the operator $A(t)$ but in change it requires the existence of a sufficiently regular solution X for equation (1.1)

$(A(t)x \in L^2((0, T) \times \Omega; H))$ which is not the case for Examples 4.1, 4.2. Such a condition holds, however, for the stochastic reaction-diffusion equation

$$\begin{aligned} dX - \Delta X \, dt + \Psi(X)dt &= X \, dW \text{ in } (0, T) \times \mathcal{O}, \\ X &= 0 \text{ on } (0, T) \times \partial\mathcal{O}, \\ X(0) &= x, \end{aligned}$$

if $x \in H_0^1(\mathcal{O})$ and $\Psi : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and monotonically increasing and for other stochastic parabolic equations as well.

Acknowledgments. Financial support through SFB 701 at Bielefeld University is gratefully acknowledged.

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